



TITLE:

# ON UNIVALENT FUNCTIONS WITH HALF-INTEGER COEFFICIENTS (Conditions for Univalence of Functions and Applications)

AUTHOR(S):

HIRANUMA, NAOKI

---

CITATION:

HIRANUMA, NAOKI. ON UNIVALENT FUNCTIONS WITH HALF-INTEGER COEFFICIENTS (Conditions for Univalence of Functions and Applications). 数理解析研究所講究録 2011, 1772: 18-20

ISSUE DATE:

2011-12

URL:

<http://hdl.handle.net/2433/171693>

RIGHT:

# ON UNIVALENT FUNCTIONS WITH HALF-INTEGER COEFFICIENTS

NAOKI HIRANUMA

ABSTRACT. Let  $\mathcal{S}$  be the class of functions  $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$  which are analytic and univalent in the unit disk  $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ . The subclasses of  $\mathcal{S}$  whose coefficients  $a_n$  belong to a quadratic field have been studied by Friedman [3] and Bernardi [1]. Linis [7] gave a short proof of Friedman's theorem which states that if all the  $a_n$  are "rational integers" then  $f$  is rational and has nine forms. In this paper, we consider what will happen if all the  $a_n$  are "half-integers"; that is,  $2a_n \in \mathbb{Z}$ .

## 1. PRELIMINARIES

**1.1. Notation and Definitions.** A *domain* is an open connected set in the complex plane  $\mathbb{C}$ . The *unit disk*  $\mathbb{D}$  consists of all points  $z \in \mathbb{C}$  of modulus  $|z| < 1$ . A single-valued function  $f$  is said to be *univalent* in a domain  $D \subset \mathbb{C}$  if it is injective; that is, if  $f(z_1) \neq f(z_2)$  for all points  $z_1$  and  $z_2$  in  $D$  with  $z_1 \neq z_2$ . The function  $f$  is said to be *locally univalent* at a point  $z_0 \in D$  if it is univalent in some neighborhood of  $z_0$ . For analytic functions  $f$ , the condition  $f'(z_0) \neq 0$  is equivalent to local univalence at  $z_0$ .

We shall be concerned primarily with the class  $\mathcal{S}$  of functions  $f$  analytic and univalent in  $\mathbb{D}$ , normalized by the conditions  $f(0) = 0$  and  $f'(0) = 1$ . Thus each  $f \in \mathcal{S}$  has a Taylor series expansion of the form

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots, \quad |z| < 1.$$

The important example of a function in the class  $\mathcal{S}$  is the *Koebe function*

$$k(z) = \frac{z}{(1-z)^2} = z + 2z^2 + 3z^3 + \cdots.$$

**1.2. Bieberbach's Conjecture.** In 1916, Bieberbach estimated the second coefficient  $a_2$  of a function in the class  $\mathcal{S}$ . (See [2, p. 30].)

**Theorem 1.** *If  $f \in \mathcal{S}$  then  $|a_2| \leq 2$ . Equality occurs if and only if  $f$  is the Koebe function or one of its rotations.*

This suggests the general problem to find

$$A_n := \sup_{f \in \mathcal{S}} |a_n|, \quad n = 2, 3, \dots$$

In a footnote, he wrote "Vielleicht ist überhaupt  $A_n = n$  (Perhaps it is generally  $A_n = n$ ). Since the Koebe function plays the extremal role in so many problems for the class  $\mathcal{S}$ , it is natural to suspect that it maximizes  $|a_n|$  for all  $n$ . This is the famous conjecture of Bieberbach, first proposed in 1916.

Many partial results were obtained in the intervening years, including results for special subclasses of  $\mathcal{S}$  and for particular coefficients, as well as asymptotic estimates and estimates for general  $n$ . Finally, de Branges [4] gave a remarkable proof in 1985. (See [6].)

**Theorem 2.** If  $f \in \mathcal{S}$  then

$$|a_n| \leq n, \quad n = 2, 3, \dots \quad (1)$$

Equality occurs if and only if  $f$  is the Koebe function or one of its rotations.

**1.3. Prawitz' Inequality.** Let  $f \in \mathcal{S}$ . Set  $F(z) = z/f(z) = \sum_{n=0}^{\infty} b_n z^n$ , then

$$F(z) = 1 - a_2 z + (a_2^2 - a_3) z^2 + \dots$$

Hence, we have  $b_0 = 1$ ,  $b_1 = -a_2$ ,  $b_2 = a_2^2 - a_3, \dots$ . The coefficient  $b_n$  ( $n \geq 1$ ) can be computed by the relation

$$b_n = (-1)^n \begin{vmatrix} a_2 & 1 & \dots & 0 \\ a_3 & a_2 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ a_{n+1} & a_n & \dots & a_2 \end{vmatrix}.$$

Prawitz [8] discovered an estimate for the coefficient  $b_n$ . It is a generalization of the Gronwall area theorem (see [2, p. 29]) and may be formulated as follows:

**Theorem 3.** Let  $f \in \mathcal{S}$  and  $[z/f(z)]^{\alpha/2} = \sum_{n=0}^{\infty} \beta_n z^n$ . Then

$$\sum_{n=0}^{\infty} \frac{(2n - \alpha)}{\alpha} |\beta_n|^2 \leq 1$$

for all real  $\alpha$ .

In particular, for  $\alpha = 2$  we have the following

**Corollary 1.** Let  $f \in \mathcal{S}$  and  $z/f(z) = \sum_{n=0}^{\infty} b_n z^n$ . Then

$$\sum_{n=1}^{\infty} (n-1) |b_n|^2 \leq 1. \quad (2)$$

This corollary is essentially equivalent to the Gronwall area theorem.

## 2. MOTIVATION

**2.1. Friedman's Theorem.** Friedman [3] proved the following theorem which is a part of Salem's theorem on univalent functions [10]:

**Theorem 4.** Let  $f \in \mathcal{S}$ . If all the coefficients  $a_n$  are rational integers then  $f(z)$  is one of the following nine functions:

$$z, \quad \frac{z}{1 \pm z}, \quad \frac{z}{1 \pm z^2}, \quad \frac{z}{(1 \pm z)^2}, \quad \frac{z}{1 \pm z + z^2}.$$

*Proof.* Set  $F(z) = z/f(z) = 1 + \sum_{n=1}^{\infty} b_n z^n$ , then the coefficients  $b_n$  are rational integers. Since  $b_1 = -a_2$  and  $|a_2| \leq 2$ , it follows that  $|b_1| \leq 2$ . Applying the inequality (2), we have  $|b_2| \leq 1$  and  $b_n = 0$  for  $n \geq 3$ . Therefore, the possible values for  $b_n$  are:

$$b_1 = 0, \pm 1, \pm 2; \quad b_2 = 0, \pm 1; \quad b_n = 0 \text{ for } n \geq 3.$$

From the combination of these values we obtain 15 functions. However, the following six functions must be rejected as having zeros in  $\mathbb{D}$ :

$$1 \pm 2z, \quad 1 \pm 2z - z^2, \quad 1 \pm z - z^2.$$

The remaining nine functions prove the theorem. □

**2.2. Extensions of Friedman's Theorem.** The method of the proof of Friedman's theorem in the previous section was given by Linis [7]. He also proved the following

**Theorem 5.** *Let  $f \in \mathcal{S}$ . If all the coefficients  $a_n$  are Gaussian integers then  $f$  has 15 forms. Here, a Gaussian integer is a complex number whose real and imaginary part are both rational integers.*

Royster [9] extended the method of the proof given by Linis to quadratic fields with negative discriminant as follows:

**Theorem 6.** *Let  $f \in \mathcal{S}$ . If all the coefficients  $a_n$  are algebraic integers in the quadratic field  $\mathbb{Q}(\sqrt{d})$  for some square-free rational negative integer  $d$ , then  $f$  has 36 forms.*

As mentioned above, they have obtained new results by replacing the condition "rational integers" with other conditions.

### 3. MAIN RESULT

**3.1. Subclass of  $\mathcal{S}$  Having Half-integer Coefficients.** Now, we shall consider what will happen if all the coefficients  $a_n$  are half-integers. Here,  $a_n$  is said to be a *half-integer* if  $2a_n$  is a rational integer.

In a similar way used in the proof of Friedman's theorem in the second chapter, we set  $F(z) = z/f(z) = 1 + \sum_{n=1}^{\infty} b_n z^n$ , then the coefficients  $b_n$  are rational numbers. In the case when the  $a_n$  are rational integers, we could obtain all the possible values for the  $b_n$ . But, in this case we cannot obtain them immediately. However, using the inequalities (1) and (2), we can examine the possibilities of coefficients one by one, and obtain the following

**Theorem 7.** *Let  $f \in \mathcal{S}$ . If all the coefficients  $a_n$  are half-integers then  $f(z)$  is one of the following 13 functions:*

$$z, \quad z \pm \frac{1}{2}z^2, \quad \frac{z}{1 \pm z}, \quad \frac{z}{1 \pm z^2}, \quad \frac{z}{(1 \pm z)^2}, \quad \frac{z}{1 \pm z + z^2}, \quad \frac{z(2 \pm z)}{2(1 \pm z)^2}.$$

The detailed proof of this theorem is given in [5].

### REFERENCES

- [1] S. Bernardi, *Two theorems on schlicht functions*, Duke Math. J. **19** (1952). 5–21.
- [2] P. L. Duren, *Univalent Functions*, Springer-Verlag, New York (1983).
- [3] B. Friedman, *Two theorems on schlicht functions*, Duke Math. J. **13** (1946). 171–177.
- [4] L. de Branges, *A proof of the Bieberbach conjecture*, Acta Math. **154** (1985), no. 1-2, 137–152.
- [5] N. Hiranuma, *On univalent functions with half-integer coefficients*, in preparation (2011).
- [6] W. Koepf, *Bieberbach's conjecture, the de Branges and Weinstein functions and the Askey-Gasper inequality* (English summary), Ramanujan J. **13** (2007). 103–129.
- [7] V. Linis, *Note on univalent functions*, Amer. Math. Monthly **62** (1955). 109–110.
- [8] H. Prawitz, *Über mittelwerte analytischer funktionen* (German), Arkiv for Mat., Astr., och Fysik, **20**, no. 6 (1927). 1–12.
- [9] W. C. Royster, *Rational univalent functions*, Amer. Math. Monthly **63** (1956). 326–328.
- [10] R. Salem, *Power series with integral coefficients*, Duke Math. J. **12** (1945). 153–172.

DIVISION OF MATHEMATICS  
GRADUATE SCHOOL OF INFORMATION SCIENCES  
TOHOKU UNIVERSITY  
6-3-09 ARAMAKI-AZA-AOBA, AOBA-KU, SENDAI 980-8579, JAPAN  
E-mail address: hiranumanaoki@gmail.com